ELSEVIER

Topology and its Applications 74 (1996) 283–298

TOPOLOGY AND ITS APPLICATIONS

# Cyclic monotonically normal spaces from Cantor sets

Akio Kato [1]

*Department of Mathematics, National Defense Academy, Yokosuka, 239 Japan*

Received 24 August 1995; revised 16 February 1996

**Abstract**

We generalize M.E. Rudin's construction in a geometric way to produce various non-acyclic, monotonically normal spaces from Cantor sets.

*Keywords:* Monotonically normal space; Perfectly normal space

*AMS classification*: 54A10; 54D15; 54G20

## 1. Monotonically normal ternary example

M.E. Rudin [14] constructed a non-acyclic, monotonically normal space using a ternary tree. We will simplify (and generalize in Section 2) her construction in a geometric way. Let $(X,\tau)$ be a topological space with a topology $\tau$. Throughout this paper, we assume that all spaces are at least Hausdorff. Let $H$ be an operation defined for every pair $(x,U)$ with $x \in U \in \tau$ such that $H(x,U)$ is an open neighborhood of $x$ included in $U$. $H$ is called *monotone* if it "respects" the inclusion, that is, $x \in U \subseteq V$ implies $H(x,U) \subseteq H(x,V)$. An obvious example of a monotone operator is the $H$ such that $H(x,U) = U$. A sequence $\{x_i\}_{i<n}$ of distinct points in $X$ is called an *n-cycle* of $H$ if

$$\bigcap_{i<n} H(x_i, X \setminus \{x_{i-1}\}) \neq \emptyset, \quad \text{where } x_{-1} \text{ means } x_{n-1}.$$

A space $X$ is called *monotonically normal* if it has a monotone operator $H$ without any 2-cycles, that is,

$$H(x, X \setminus \{y\}) \cap H(y, X \setminus \{x\}) = \emptyset$$

[1] E-mail: akiokato@cc.nda.ac.jp.


*SSDI* 0166-8641(96)00062-4

for any distinct two points $x$, $y$ in $X$. Further, if $X$ has a monotone operator without any cycles, $X$ is called *acyclic monotonically normal*. For monotonic normality and related topics see [6,8,9]. Let us consider the ternary Cantor set $3^\omega = \{0,1,2\}^\omega$. Let $P = 3^{<\omega} = \bigcup\{3^n\colon n \in \omega\}$ be the set of all finite sequences of 0, 1, 2. For each $s \in P$, put $B(s) = \{x \in 3^\omega\colon s \subset x\}$ and $L(s) = B(s^+)$ where $s^+ = t\hat{}(i \dotplus 1)$ if $s = t\hat{}i$. Here $\dotplus$ means the addition modulo 3, and $t\hat{}j$ denotes $t \cup \{(n,j)\}$ if $t \in 3^n$. We make a convention that $L(\emptyset) = \emptyset$. "B" stands for "ball" or "box", while "L" means the "left" of $B(s)$. For $x \in 3^\omega$ and $s \in P$ with $s \subset x$, define

$$L(x;s) = \{x\} \cup \bigcup \bigl\{L(t)\colon\ t \in P \text{ and } s \subseteq t \subset x\bigr\}, \quad \text{and} \quad L(x) = L(x;\emptyset).$$

The collection

$$\mathbb{L} = \bigl\{L(x;s)\colon\ x \in 3^\omega,\ s \in 3^{<\omega} \text{ with } s \subset x\bigr\} = \bigl\{L(x; x \restriction m)\colon\ x \in 3^\omega,\ m \in \omega\bigr\}$$

generates a new topology on $3^\omega$, which we will denote by $\tau_{3,1}$ (this notation is a special case of $\tau_{p,q}$ in Section 2). The standard Cantor topology on $3^\omega$ generated by $B(s)$ ($s \in P$) is denoted by $\tau_3$. Note that each $L(x;s)$ is closed with respect to $\tau_3$ and $L(x;s) \setminus \{x\}$ is clopen in $3^\omega \setminus \{x\}$ with respect to $\tau_3$. Hence the collection $\mathbb{L}$ forms a clopen base for $\tau_{3,1}$. And so, $(3^\omega, \tau_{3,1})$ is a 0-dimensional Hausdorff space. In terms of generalized metric theory, this space is *cometrizable* in the sense that it has a weaker metric topology such that each point has a neighborhood base consisting of sets closed in the metric topology (cf. [7]) . The shaded portion of Fig. 1 shows what a neighborhood of the point $x = (0,1,2,\ldots) \in 3^\omega$ looks like with respect to the topology $\tau_{3,1}$. We note that the space $(3^\omega, \tau_{3,1})$ is the same as the Rudin's example [14] except that her example adds countably many isolated points. For every pair $(x, U)$ with $x \in U \in \tau_{3,1}$ define $H_{3,1}(x,U)$ to be the maximal $L(x;s)$ contained in $U$. Clearly, this operator $H_{3,1}$ is monotone.

Rudin proved

**Theorem 1.0** (Rudin [14]). (1) *$H_{3,1}$ does not have any 2-cycles. Hence $(3^\omega, \tau_{3,1})$ is monotonically normal.*

(2) *Every monotone operator for $(3^\omega, \tau_{3,1})$ has a 3-cycle. Hence $(3^\omega, \tau_{3,1})$ is not acyclic monotonically normal.*

For a geometric proof of this theorem, see the special case $p = 3$, $q = 1$ of Theorems 2.3 and 2.5 below. As for homogeneity, it is easy to check that our space $(3^\omega, \tau_{3,1})$ is homogeneous. We can say more: indeed, every clopen subset is homeomorphic to the whole space. To prove this kind of fact and for also later purpose we introduce the following terminology. A 0-dimensional space $Z$ is called *clopen-homogeneous* if, for every pair of proper clopen subsets $U$, $V$ of $Z$ there exists an autohomeomorphism of $Z$ which maps $U$ onto $V$; note that this is equivalent to say simply that all proper clopen subsets are homeomorphic. In [12] the term "$h$-homogeneity" is used for a space $Z$ if every proper clopen subset is homeomorphic to the whole space $Z$. These two notions coincide for nonpseudocompact spaces as the next lemma shows. The symbol $\approx$ means "is homeomorphic to".

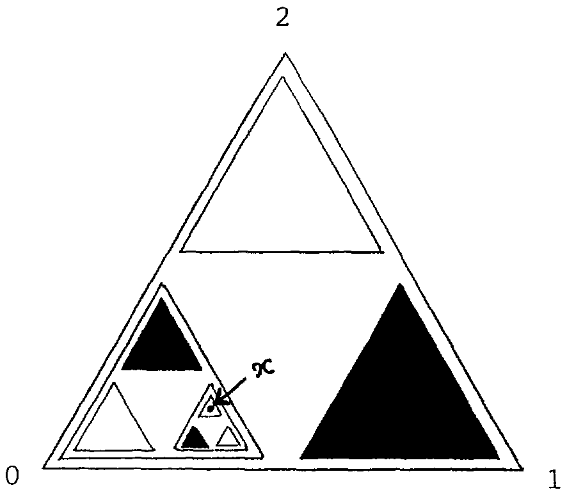


Fig. 1.

**Lemma 1.1.** *Let $Z$ be a 0-dimensional nonpseudocompact space. Then $Z$ is clopen-homogeneous iff every proper clopen subset is homeomorphic to the whole space $Z \approx Z \times \omega$.*

**Proof.** We need to prove the "only if" part. Suppose a 0-dimensional nonpseudocompact space $Z$ is clopen-homogeneous. Then $Z$ can be decomposed as $Z \approx U \times \omega$ for some clopen $U$. Since $U \times \omega$ is decomposed as a topological sum of $U$ and $U \times \omega$, we can see that $U$ and $U \times \omega$ are both proper clopen subset of $Z$; hence $U \approx U \times \omega$. Consequently, $Z \approx U$ and $Z \approx U \times \omega \approx Z \times \omega$. □

Motorov [12] and Terada [17] established the following easy-to-apply criterion for clopen-homogeneity. Note that clopen-homogeneity implies homogeneity if the space is 1st countable. A collection of nonempty open sets is called a *$\pi$-base* if every nonempty open subset of $Z$ contains an element of the collection.

**Fact 1.2** (Motorov [12], Terada [17]). *Let $Z$ be a 0-dimensional nonpseudocompact space. If $Z$ has a $\pi$-base consisting of clopen sets which are all homeomorphic to $Z$, then $Z$ is clopen-homogeneous.*

In $X = (3^\omega, \tau_{3,1})$, every neighborhood $L(x; s)$ $(s \neq \emptyset)$ contains $L(s) = B(s^+) \approx X$, we can apply the above criterion. Hence $X = (3^\omega, \tau_{3,1})$ is clopen-homogeneous and $X \times \omega \approx X$.

We next observe that our space $X = (3^\omega, \tau_{3,1})$ contains the familiar Sorgenfrey line. To see this, let us consider the Cantor subset $C$ naturally embedded in $X$ as $C = 2^\omega = \{0, 1\}^\omega \subset 3^\omega$. For $x, y \in C$ define the linear order: $x < y$ iff there exists some $m \in \omega$ such that $x \restriction m = y \restriction m$ and $x(m) = 0 < 1 = y(m)$. Put $C_\star = \{x \in 2^\omega \colon \forall n \in \omega\ \exists m >

$n\ x(m) = 0\}$; then $C \setminus C_\star$ is a countable set consisting of $x$ which is eventually 1. Let $f : C \to [0,1]$ be the standard function $f(x) = \sum_{n\in\omega} x(n)/2^{n+1}$ onto the unit interval. Then $f \restriction C_\star$ is an order-isomorphism from $C_\star$ onto [0,1). Hence, if we consider the right half-open interval topology on $C_\star$ with respect to $<$, then this space is identical with the Sorgenfrey line [0,1). Note that, for $x, y \in C_\star$,

$$x < y \quad \text{iff} \quad \exists s \in P\ x \in B(s\hat{\ }0) \text{ and } y \in B(s\hat{\ }1)$$
$$\text{iff} \quad \exists s \in P\ y \in L(x; s\hat{\ }0).$$

This shows that $C_\star$ with the subspace topology of $\tau_{3,1}$ is the Sorgenfrey line. Observe that $C_\star$ is closed with respect to $\tau_{3,1}$. Now, for each $i < 3$ put

$$A_i = \{x \in 3^\omega\colon\ \forall n \in \omega\ x(n) = i \text{ or } i+1, \text{ and } \forall n \in \omega\ \exists m > n\ x(m) = i\}.$$

Then $C_\star = A_0 \approx A_1 \approx A_2$. Define $A$, $B$ and $X_\star$ by $A = A_0 \cup A_1 \cup A_2$, $B = 0\hat{\ }A_1 \cup 1\hat{\ }A_2 \cup 2\hat{\ }A_0$ and $X_\star = A \cup \bigcup_{s\in P} s\hat{\ }B$. Here $i\hat{\ }A$ and $s\hat{\ }B$ stand for $\{i\hat{\ }x\colon\ x \in A\}$ and $\{s\hat{\ }x\colon\ x \in B\}$ respectively. We call $X_\star$ the *skeleton* of $X = 3^\omega$. Since $C_\star \approx A \approx B \approx s\hat{\ }B$ and each of these sets is closed with respect to $\tau_{3,1}$ we can conclude that the skeleton $X_\star$ is a disjoint union of countably many, closed Sorgenfrey lines. Summarizing what we have obtained, we get the following theorem. Recall the fact that every ccc monotonically normal space is hereditary Lindelöf [13].

**Theorem 1.3.** $X = (3^\omega, \tau_{3,1})$ *is a* 0*-dimensional,* 1*st countable, cometrizable, homogeneous space such that:*

(1) $X$ *is separable and monotonically normal, hence, hereditary Lindelöf.*

(2) $X$ *is not acyclic monotonically normal.*

(3) $X$ *is clopen-homogeneous and* $X \approx X \times \omega$.

(4) $X$ *contains the dense "skeleton"* $X_\star$ *which is the union of countably many, disjoint closed Sorgenfrey lines.*

**Remark 1.4.** Recall that a space $Y$ is called a "$K_0$-space" if every subspace $A$ of $Y$ possesses an operation

$$k_A : (\text{open sets in } A) \to (\text{open sets in } Y)$$

such that

(i) $k_A(\emptyset) = \emptyset$;

(ii) $k_A(U) \cap A = U$ for every open set $U$ in $A$;

(iii) $k_A(U \cap V) = k_A(U) \cap k_A(V)$ for every open sets $U$, $V$ in $A$.

Rudin [14] showed that her original example is not $K_0$. But we don't know if our space $(3^\omega, \tau_{3,1})$, which is a closed subspace of her example, is a $K_0$-space. This question is related to the Moody and Roscoe conjecture [11] that every monotonically normal $K_0$-space is acyclic monotonically normal.

## 2. New topologies on $p^\omega$

Let $p$ and $q$ be integers such that $0 < q < p$. We will generalize the construction of the topology $\tau_{3,1}$ in Section 1 to define a new topology $\tau_{p,q}$ on the product $p^\omega = \{0,1,\ldots,p-1\}^\omega$. Let $p^{<\omega} = \bigcup_{n\in\omega} p^n$ be the finite sequences of $0,1,\ldots,p-1$. For each $s \in p^{<\omega}$ put $B(s) = \{x \in p^\omega\colon s \subset x\}$. For each $s = u\hat{\ }i \in p^{<\omega}$ define

$$L_q(s) = \bigcup\{B(u\hat{\ }(i \dotplus j))\colon 1 \leqslant j \leqslant q\},$$

where $\dotplus$ stands for the addition modulo $p$. We make a convention that $L_q(\emptyset) = \emptyset$. For each $x \in p^\omega$ and $s \in p^{<\omega}$ with $s \subset x$, define

$$L_q(x;s) = \{x\} \cup \bigcup\{L_q(t)\colon t \in p^{<\omega},\ s \subseteq t \subset x\} \quad \text{and} \quad L_q(x) = L_q(x;\emptyset).$$

Put

$$\mathbb{L}_{p,q} = \{L_q(x;s)\colon x \in p^\omega,\ s \in p^{<\omega},\ s \subset x\} = \{L_q(x; x \restriction m)\colon x \in p^\omega,\ m \in \omega\}.$$

We denote by $\tau_{p,q}$ the topology generated by the collection $\mathbb{L}_{p,q}$. It is clear that $\tau_{p,q}$ for $p = 3$, $q = 1$ is just the topology considered in Section 1. Let us denote by $\tau_p$ the standard Cantor product topology on $p^\omega$ generated by $B(s)$ $(s \in p^{<\omega})$. Note that $\tau_{p,p-1} = \tau_p$ and hence our interest is on the case $1 \leqslant q < p-1$. It is easy to see that the collection $\mathbb{L}_{p,q}$ forms a clopen base for $\tau_{p,q}$ and that each neighborhood $L_q(x;s)$ contains a clopen set homeomorphic to the whole space $(p^\omega, \tau_{p,q})$. Hence, by the same way as in Section 1, we know

**Proposition 2.1.** *$X = (p^\omega, \tau_{p,q})$ is 1st countable, separable, cometrizable, 0-dimensional Hausdorff and homogeneous. Moreover, $X$ is clopen-homogeneous and $X \times \omega \approx X$.*

The shaded portion of Fig. 2 shows what a neighborhood of the point $x = (0,2,1,\ldots) \in 4^\omega$ looks like with respect to $\tau_{4,2}$. As is well known, all spaces $(p^\omega, \tau_p)$,

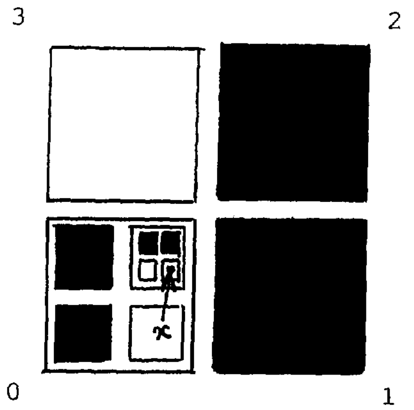


Fig. 2.

where $2 \leqslant p \in \omega$, are homeomorphic to the Cantor space $(2^\omega, \tau_2)$. Therefore, it is natural to ask if our spaces $(p^\omega, \tau_{p,q})$, where $1 \leqslant q < p-1$, are topologically distinct. A bit amazingly we can show that they are quite distinct by checking the existence of various cycles; but we could not succeed in classifying them completely into topologically distinct spaces (see Theorem 2.7 and Remark 2.8). Let us first define the "standard" monotone operator $H_{p,q}$ for $(p^\omega, \tau_{p,q})$ and investigate its properties. For every $(x, U)$ with $x \in p^\omega$ and $x \in U \in \tau_{p,q}$, let $H_{p,q}(x, U)$ be the maximal $L_q(x; s)$ contained in $U$. It is clear that this $H_{p,q}$ is monotone, and that $H_{3,1}$ in Section 1 is the special case of $p = 3$, $q = 1$. For a real number $x$ we use the symbol $\lfloor x \rfloor$ to denote the integer $n$ such that $n \leqslant x < n+1$, while the symbol $\lceil x \rceil$ denotes the integer $m$ such that $m-1 < x \leqslant m$.

**Lemma 2.2.** *Let $n$ and $p$ be natural numbers such that $2 \leqslant n < p$.*

(1) *$\lfloor (1-1/n)p \rfloor$ is equal to $p-\alpha$ if $p = n \cdot \alpha$ for some $\alpha \in \omega$, while, is equal to $p - \alpha - 1$ if $p = n \cdot \alpha + j$ for some $\alpha$, $j \in \omega$ with $0 < j < n$.*

(2) *If $1 \leqslant q < p-1$, then the condition $\lfloor (1-1/n)p \rfloor \leqslant q$ is equivalent with $n < p/(p-q-1)$. In particular, $\lfloor (1-1/n)p \rfloor \leqslant p-2$.*

(3) *$\lfloor (1-1/n)p \rfloor$ is equal to $\max\{d_0 + d_1 + \cdots + d_{n-2}\}$ where $\max$ is taken with respect to all the partitions of $p$ such that*

$$p = d_0 + d_1 + \cdots + d_{n-2} + d_{n-1}; \quad 1 \leqslant d_0 \leqslant d_1 \leqslant \cdots \leqslant d_{n-2} \leqslant d_{n-1}.$$

*(The last condition can obviously be weakened to $1 \leqslant d_0, d_1, \ldots, d_{n-2} \leqslant d_{n-1}$.)*

**Proof.** (1) and (2) are easy. Let us prove (3). Denote by $M$ the maximum value considered in the above (3). In case $p = n \cdot \alpha$ for some $\alpha \in \omega$, the maximum value $M$ is attained by the partition $p = n \cdot \alpha = \alpha + \alpha + \cdots + \alpha$. Hence $M = (n-1)\alpha = p - \alpha$. In case $p = n \cdot \alpha + j$, $0 < j < n$, the maximum value $M$ is attained by the partition $p = (n-j)\alpha + j(\alpha+1) = \alpha + \cdots + \alpha + (\alpha+1) + \cdots + (\alpha+1)$. Hence $M = p - (\alpha+1)$. Thus the calculation (1) leads to (3). $\square$

The next result generalizes Rudin's result 1.0(1).

**Theorem 2.3.** *Let $2 \leqslant n < p$. If $\lfloor (1-1/n)p \rfloor \leqslant q < p-1$, then $H_{p,q}$ does not have any $n$-cycle.*

**Proof.** Take any $n$ distinct points $\{x_i\}_{i<n}$ in $X = p^\omega$. We need to show that

$$\bigcap_{i<n} H_{p,q}(x_i, X \setminus \{x_{i-1}\}) = \emptyset, \quad \text{where } x_{-1} = x_{n-1}. \tag{I}$$

Put $A = \{x_i\}_{i<n}$. Take the minimal $B(s)$ that contains $A$, and let $j_0, j_1, \ldots, j_{m-1}$ $(2 \leqslant m \leqslant n)$ be the listing of all the distinct $0 \leqslant j < p$ such that $A \cap B(s\hat{\ }j) \neq \emptyset$. Arrange this listing as

$$j_0 \dot{+} d_0 = j_1,\ j_1 \dot{+} d_1 = j_2,\ \ldots,\ j_{m-1} \dot{+} d_{m-1} = j_0$$

where $1 \leqslant d_0, d_1, \ldots, d_{m-2} \leqslant d_{m-1}$; recall that $\dot{+}$ means the addition modulo $p$. Put $s\hat{\ }j_0 = t$, and choose $\alpha, \beta < n$ such that

$$x_\alpha, x_{\beta-1} \in B(t) \quad \text{and} \quad x_{\alpha-1}, x_\beta \notin B(t).$$

To prove (I), it suffices to show that

$$H_{p,q}\big(x_\alpha, X \setminus \{x_{\alpha-1}\}\big) \cap H_{p,q}\big(x_\beta, X \setminus \{x_{\beta-1}\}\big) = \emptyset. \tag{II}$$

By Lemma 2.2(3) we have $d_0 + \cdots + d_{m-2} \leqslant \lfloor (1-1/m)p \rfloor \leqslant \lfloor (1-1/n)p \rfloor$. Hence our condition $\lfloor (1-1/n)p \rfloor \leqslant q$ implies $d_0 + \cdots + d_{m-2} \leqslant q$. Consequently we have $x_{\alpha-1} \in L_q(x_\alpha; t)$. Therefore the definition of $H_{p,q}$ implies

$$H_{p,q}\big(x_\alpha, X \setminus \{x_{\alpha-1}\}\big) \subseteq B(t).$$

On the other hand, the conditions $x_\beta \notin B(t)$, $x_{\beta-1} \in B(t)$ imply

$$H_{p,q}\big(x_\beta, X \setminus \{x_{\beta-1}\}\big) \cap B(t) = \emptyset.$$

Thus we get (II). □

**Corollary 2.4.** *If $\lfloor p/2 \rfloor \leqslant q < p-1$, then $H_{p,q}$ does not have any 2-cycle; consequently, $(p^\omega, \tau_{p,q})$ is monotonically normal.*

This corollary tells, for example, that $\tau_{3,1}$, $\tau_{4,2}$, $\tau_{5,2}$ and $\tau_{5,3}$ are monotonically normal. The next theorem generalizes Rudin's result 1.0(2), showing that no $\tau_{p,q}$ is acyclic monotonically normal.

**Theorem 2.5.** *Let $1 \leqslant q < p-1$, and let $G$ be an arbitrary monotone operator for $(p^\omega, \tau_{p,q})$. Then $G$ has a $p$-cycle $x_0, x_1, \ldots, x_{p-1} \in p^\omega$ satisfying*

$$L_q(t) \subseteq \bigcap_{i<p} G\big(x_i, L_q(x_i)\big) \quad \textit{and} \quad x_i \in B(t\hat{\ }i)\ (i<p) \tag{$\star$}$$

*for some $\emptyset \neq t \in p^{<\omega}$.*

**Proof.** Put $X = p^\omega$. Note first that the above condition $(\star)$ implies that $\{x_i\}_{i<p}$ forms a $p$-cycle for $G$. Indeed, the conditions $q < p-1$ and $x_i \in B(t\hat{\ }i)$ imply that $L_q(x_i)$ is included in $X \setminus \{x_{i-1}\}$ (note $x_{-1} = x_{p-1}$). Hence, by the monotonicity of $G$, we have

$$G\big(x_i, L_q(x_i)\big) \subseteq G\big(x_i, X \setminus \{x_{i-1}\}\big).$$

Consequently $(\star)$ implies $\bigcap_{i<p} G(x_i, X \setminus \{x_{i-1}\}) \neq \emptyset$.

Now we show how to choose the points $x_i$ $(i<p)$ with $(\star)$. Let $x \in X$ be an arbitrary point. Since $G(x, L_q(x))$ is a neighborhood of $x$ with respect to $\tau_{p,q}$, we can choose $\varphi(x) \in p^{<\omega}$ such that

$$\varphi(x) \subset x \quad \text{and} \quad L_q\big(x; \varphi(x)\big) \subseteq G\big(x, L_q(x)\big).$$

Consider the mapping $\varphi : X \to p^{<\omega}$. Then

$$X = \bigcup\{\varphi^{-1}(s)\colon\ s \in p^{<\omega}\}, \quad \varphi^{-1}(s) \subseteq B(s).$$

Since $p^{<\omega}$ is countable and the space $X$ with the topology $\tau_p$ is homeomorphic with the Cantor space, we can apply the Baire Category Theorem to find $u \in p^{<\omega}$ such that $\operatorname{int}\operatorname{cl}\varphi^{-1}(u) \neq \emptyset$ with respect to $\tau_p$, that is, there exists $\emptyset \neq t \in p^{<\omega}$ with

$$B(t) \subseteq \operatorname{cl}\varphi^{-1}(u) \subseteq B(u) \quad (\text{"cl" is with respect to } \tau_p).$$

Since $B(t\hat{}i) \subseteq \operatorname{cl}\ \varphi^{-1}(u)$ for each $i < p$, we can select $x_i \in B(t\hat{}i) \cap \varphi^{-1}(u)$. The condition $x_i \in \varphi^{-1}(u)$ implies $L_q(x_i; u) \subseteq G(x_i; L_q(x_i))$. Hence

$$L_q(t) \subseteq L_q(x_i; t) \subseteq L_q(x_i; u) \subseteq G(x_i, L_q(x_i)),$$

which shows $L_q(t) \subseteq \bigcap_{i<p} G(x_i, L_q(x_i))$. □

Comparing 2.3 and 2.5, we know that a large cycle does not always contain a small subcycle. As for subcycle of the $p$-cycle in 2.5 we can assert the following.

**Lemma 2.6.** *Let $2 \leqslant n < p$ and let everything be as in 2.5. If $q < \lfloor(1 - 1/n)p\rfloor$, then the $p$-cycle $\{x_i\}_{i<p}$ in 2.5 contains an $n$-cycle.*

**Proof.** Let $p = n\alpha + j$ where $\alpha, j \in \omega$ and $0 \leqslant j < n$. Note $\alpha \geqslant 1$ since $n < p$. Partition $p$ into $n$ blocks according to the formula

$$p = (n - j)\alpha + j(\alpha + 1) = \alpha + \cdots + \alpha + (\alpha + 1) + \cdots + (\alpha + 1).$$

Choose the first number from each block and list all of them as $i(0) < i(1) < \cdots < i(n-1)$. Put $x_{i(k)} = y_k$. We show that $\{y_k\}_{k<n}$ forms an $n$-cycle for $G$. Our condition $q < \lfloor(1 - 1/n)p\rfloor$ with 2.2(1) implies that $q < p - \alpha$ if $j = 0$, while $q < p - (\alpha + 1)$ if $j > 0$. Hence $L_q(y_k) \subseteq X \setminus \{y_{k-1}\}$ for each $k < n$, where $y_{-1} = y_{n-1}$. Therefore, by the $(\star)$ in 2.5 we get

$$L_q(t) \subseteq \bigcap_{k<n} G(y_k, L_q(y_k)) \subseteq \bigcap_{k<n} G(y_k, X \setminus \{y_{k-1}\}),$$

showing that $\{y_k\}_{k<n}$ forms an $n$-cycle. □

Define the *cyclicity number* $\operatorname{Cyc}(\tau_{p,q})$ of $\tau_{p,q}$ as the minimal number $2 \leqslant k \in \omega$ such that every monotone operator for $\tau_{p,q}$ has a $k$-cycle. Assembling the hitherto results we can completely determine $\operatorname{Cyc}(\tau_{p,q})$ (see Theorem 2.7). Note that $\tau_{p,q}$ is monotonically normal iff it has a monotone operator without any 2-cycle, that is, $\operatorname{Cyc}(\tau_{p,q}) \geqslant 3$. Theorem 2.5 shows that $2 \leqslant \operatorname{Cyc}(\tau_{p,q}) \leqslant p$. Theorem 2.3 combined with 2.2(2) implies that if $n < p/(p-q-1)$ then $\operatorname{Cyc}(\tau_{p,q}) > n$. On the other hand, 2.6 with 2.2(2) shows that if $n \geqslant p/(p - q - 1)$ then $\operatorname{Cyc}(\tau_{p,q}) \leqslant n$. Thus we get the following conclusion. Recall that for a real number $x$ the symbol $\lceil x\rceil$ means the integer $m$ such that $m - 1 < x \leqslant m$.

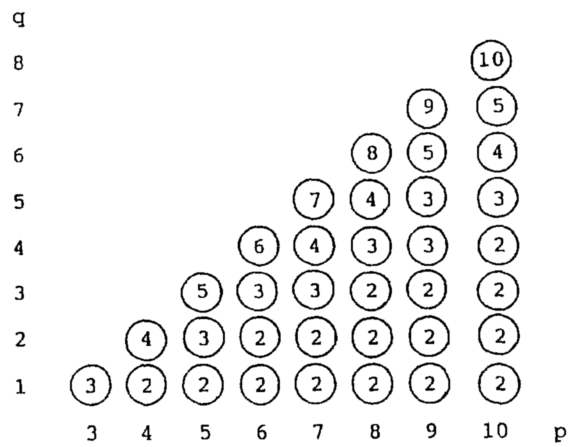


Diagram 1.

**Theorem 2.7.** *Let $1 \leqslant q < p - 1$. Then*

$$\mathrm{Cyc}(\tau_{p,q}) = \lceil p/(p - q - 1)\rceil.$$

For example $\mathrm{Cyc}(\tau_{p,p-2}) = p$ for $p \geqslant 3$, and $\mathrm{Cyc}(\tau_{p,p-3}) = \lceil p/2\rceil$ for $p \geqslant 4$. Diagram 1 shows the results of calculations of $\mathrm{Cyc}(\tau_{p,q})$ for $p \leqslant 10$.

Note that $\lceil p/(p - q - 1)\rceil \geqslant 3$ iff $(p - 1)/2 \leqslant q$ iff $\lfloor p/2\rfloor \leqslant q$. Therefore we get the following corollary which strengthens 2.4 and characterizes the monotonical normality of $\tau_{p,q}$.

**Corollary 2.8.** *Let $1 \leqslant q < p-1$. The topology $\tau_{p,q}$ is monotonically normal iff $\lfloor p/2\rfloor \leqslant q$ iff $(p - 1)/2 \leqslant q$.*

From this corollary we know that, if $q < \lfloor p/2\rfloor$, $\tau_{p,q}$ is not monotonically normal. But in this case we can even assert that $\tau_{p,q}$ is not normal:

**Proposition 2.9.** *Let $1 \leqslant q < \lfloor p/2\rfloor$. Then $\tau_{p,q}$ contains a closed discrete subspace of cardinality of the continuum. Hence $\tau_{p,q}$, being separable, is not normal.*

**Proof.** Put $q + 1 = d$. Define $D$ to be a subset of $p^\omega$ consisting of points $x$ such that $x(n) = 0$ or $d$ for all $n \in \omega$. Since $D$ is closed with respect to the Cantor topology $\tau_p$ on $p^\omega$, it is also closed with respect to $\tau_{p,q}$. So, we need only show that $D$ is discrete with respect to $\tau_{p,q}$. Take any $x \in D$ and consider its neighborhood

$$L_q(x) = \{x\} \cup \bigcup \{L_q(s)\colon\ s \in p^{<\omega},\ \emptyset \neq s \subset x\}.$$

It suffices to show $L_q(x) \cap D = \{x\}$. Let $s \in p^{<\omega}$ be such that $\emptyset \neq s \subset x$. Then $s = t\hat{\ }i$ where $i = 0$ or $d$. Our condition $1 \leqslant q < \lfloor p/2 \rfloor$ implies $q < d < d + q < p$. Therefore we have

$$L_q(s) \cap \big(B(t\hat{\ }0) \cup B(t\hat{\ }d)\big) = \emptyset.$$

Since $L_q(s) \subseteq B(t)$ and $D \cap B(t) \subseteq B(t\hat{\ }0) \cup B(t\hat{\ }d)$, we get $L_q(s) \cap D = \emptyset$. Thus we conclude that $L_q(x) \cap D = \{x\}$. $\square$

**Remark 2.10.** Since the cyclicity number is a topological index, spaces with different cyclicity numbers are not homeomorphic. For example, $(3^\omega, \tau_{3,1})$, $(4^\omega, \tau_{4,2})$ and $(5^\omega, \tau_{5,3})$ are not homeomorphic each other. But to distinguish all $\tau_{p,q}$, $(1 \leqslant q < p-1)$ completely, the cyclicity number itself is not enough. For instance we don't know yet if $(3^\omega, \tau_{3,1})$ is homeomorphic with $(5^\omega, \tau_{5,2})$ as they have the same cyclicity number 3.

## 3. Generalization

Let $A$ be a nonempty subset of $\{1, 2, \ldots, p-1\}$. Then we can define a topology $\tau_p(A)$ on $p^\omega$ as follows. For $s = t\hat{\ }i \in p^{<\omega}$ put

$$L_A(s) = \bigcup \big\{B\big(t\hat{\ }(i \dotplus j)\big)\colon\ j \in A\big\},$$

where $\dotplus$ means the addition modulo $p$. For $s = \emptyset$ we put $L_A(s) = \emptyset$. For $x \in p^\omega$ and $s \in p^{<\omega}$ with $s \subset x$, we define

$$L_A(x; s) = \{x\} \cup \bigcup \big\{L_A(t)\colon\ t \in p^{<\omega},\ s \subseteq t \subset x\big\} \quad \text{and} \quad L_A(x) = L_A(x; \emptyset).$$

We denote by $\tau_p(A)$ the topology generated by all sets of the form $L_A(x; s)$. Note that $\tau_{p,q}$ in Section 2 is just the case $A = \{1, 2, \ldots, q\}$. It is easy to see that conclusions of 2.1 remain intact for our general space $(p^\omega, \tau_p(A))$. Consider the natural group structure on $p^\omega$ with the additive group operation "$+$" which is induced from the modulo $p$ addition $\dotplus$ on $p$. Then our topology $\tau_p(A)$ can be viewed in a simple way as follows. For any subset $S \subseteq p^\omega$ and a point $x \in p^\omega$, we denote by $S + x$ the subset $\{z + x\colon\ z \in S\}$. Then

$$L_A(x; x \restriction m) = L_A(\mathbf{0}; \mathbf{0} \restriction m) + x,$$

where $\mathbf{0} = (0, 0, \ldots)$. Therefore the topology $\tau_p(A)$ is completely determined by the neighborhood base $L_A(\mathbf{0}; \mathbf{0} \restriction m)$ $(m \in \omega)$ at $\mathbf{0}$ and the translations "$+x$" $(x \in p^\omega)$. This observation reveals the homogeneous structure of $(p^\omega, \tau_p(A))$. Indeed, for every $x \in p^\omega$ the translation "$+x$" is an autohomeomorphism of $(p^\omega, \tau_p(A))$ since

$$\begin{aligned} L_A(y; y \restriction m) + x &= \big(L_A(\mathbf{0}; \mathbf{0} \restriction m) + y\big) + x = L_A(\mathbf{0}; \mathbf{0} \restriction m) + (y + x) \\ &= L_A\big(y + x; (y + x) \restriction m\big). \end{aligned}$$

Especially, the translation "$+(x - y)$" is an autohomeomorphism which moves the point y to the point $x$. But note that the operation $+ : (x, y) \to x + y$ is not jointly continuous with respect to $\tau_p(A)$. So, $(p^\omega, \tau_p(A), +)$ would be called a semitopological semigroup (cf. [3,15]). Let $\varphi$ be the inverse operation $\varphi(x) = -x$, i.e., $\varphi(x)(n) = p - x(n)$. Though this $\varphi$ is obviously not continuous with respect to $\tau_p(A)$, it induces a homeomorphism

$$\varphi : (p^\omega, \tau_p(A)) \to (p^\omega, \tau_p(-A)),$$

where $-A = \{p - i\colon i \in A\}$. This observation is useful: as a special case, we see

$$\tau_p(\{q+1, \ldots, p-1\}) \approx \tau_p(\{1, 2, \ldots, p-q-1\}) = \tau_{p,p-q-1}.$$

So, for instance, we have

$$\tau_3(\{2\}) \approx \tau_{3,1} \quad \text{and} \quad \tau_4(\{2,3\}) \approx \tau_{4,2}.$$

**Remark 3.1.** When $A$ is a proper subset of $\{1, 2, \ldots, p-1\}$, the pair $(\tau_p(A), \tau_p)$ is what is called a "butterfly pair" in the paper of Burke and van Douwen [2]. In particular, if $1 \leqslant q \leqslant p-2$, then $(\tau_{p,q}, \tau_p)$ is a butterfly pair. We thank the referee for pointing out the source [2].

## 4. Duplication

We here define a kind of "duplication" to obtain interesting new topological spaces, especially compact spaces. Our method corresponds to the Fedorchuk's "resolution" that resolves each point into two points (cf. [19]). Suppose a topological space $(X, \sigma)$ satisfies the following condition:

($\star$) Every point $x \in X$ has a neighborhood $U(x) \in \sigma$ which splits as $U(x) = U^1(x) \cup U^2(x)$, $U^1(x) \cap U^2(x) = \{x\}$, and $U^1(x) \setminus \{x\}$, $U^2(x) \setminus \{x\}$ are both clopen in $(U(x) \setminus \{x\}, \sigma)$.

For $i = 1, 2$ let $\sigma_i$ be the topology on $X$ generated by $\sigma \cup \{U^i(x)\colon x \in X\}$. We then call this situation that $\sigma_1$ and $\sigma_2$ are *complementary with respect to* $\sigma$. Now make disjoint copies of $X$:

$$X^+ = \{x^+\colon x \in X\}, \qquad X^- = \{x^-\colon x \in X\}$$

and put $X^\pm = X^+ \cup X^-$. (Precisely, $X^\pm = X \times \{+1, -1\}$, $X^+ = X \times \{+1\}$, $X^- = X \times \{-1\}$, $x^+ = (x, +1)$, $x^- = (x, -1)$.) For a subset $S$ of $X$ we use the following notations:

$$S^+ = \{x^+\colon x \in S\}, \qquad S^- = \{x^-\colon x \in S\}, \qquad S^\pm = S^+ \cup S^-.$$

In case $x \in S \subseteq X$, we define

$$W(x^+; S) = S^+ \cup (S \setminus \{x\})^- = S^\pm \setminus \{x^-\}, \quad \text{and}$$

$$W(x^-; S) = S^- \cup (S \setminus \{x\})^+ = S^\pm \setminus \{x^+\}.$$

We denote by $\sigma_1 \bigoplus \sigma_2$ the topology on $X^\pm$ generated by all sets of the form

$$W(x^+; U^1(x) \cap V(x)), \qquad W(x^-; U^2(x) \cap V(x)),$$

where $V(x)$ is an arbitrary neighborhood of $x$ in $\sigma$ and $U^1(x)$, $U^2(x)$ are as in the above ($\star$). We call this topology on $X^\pm$ the *duplicate* topology determined by the complementary topologies $\sigma_1$ and $\sigma_2$. Note that the subspaces $X^+$, $X^-$ in $(X^\pm, \sigma_1 \bigoplus \sigma_2)$ are homeomorphic with $(X, \sigma_1)$, $(X, \sigma_2)$ respectively. Note also that $\sigma_1 \bigoplus \sigma_2$ is Hausdorff because

$$W(x^+; S) \cap W(x^-; T) = \emptyset \quad \text{if } S \cap T = \{x\}.$$

Define the natural 2–1 map

$$\pi : \left(X^\pm, \sigma_1 \bigoplus \sigma_2\right) \to (X, \sigma)$$

by $\pi^{-1}(x) = \{x^+, x^-\}$, and observe that this is a perfect continuous map. Hence, especially, $(X^\pm, \sigma_1 \bigoplus \sigma_2)$ is compact if $(X, \sigma)$ is compact.

**Example 4.1.** For any space $(X, \sigma)$ consider the discrete topology $\sigma_d$ on $X$. Then $\sigma$ and $\sigma_d$ are complementary with respect to $\sigma$, and the resultant space $(X^\pm, \sigma \bigoplus \sigma_d)$ is the one called "Alexandroff duplicate".

**Example 4.2.** Let $(I, \sigma)$ be the unit interval $I = [0, 1]$ with the usual interval topology $\sigma$. For $x \in U(x) = I$ consider the splitting as $U^1(x) = [0, x]$, $U^2(x) = [x, 1]$. Then the corresponding topologies $\sigma_1$ and $\sigma_2$ are both the Sorgenfrey line topology, and the compact space $(I^\pm, \sigma_1 \bigoplus \sigma_2)$ is the lexicographically ordered space.

**Example 4.3.** Let $X$ be a subset of the plane $\mathbb{R}^2$ with the Euclidean subspace topology $\sigma$ such that

$$X = \left([0, 1] \times \{0\}\right) \cup \left\{\left(k/2^n, 1/2^n\right):\ k, n \in \omega \text{ and } 0 \leqslant k \leqslant 2^n\right\}.$$

Let $z \in X$. In case $z = (k/2^n, 1/2^n)$, i.e., $z$ is isolated in $X$, define trivially $U(z) = U^1(z) = U^2(z) = \{z\}$. In case $z = (t, 0)$ for some $0 \leqslant t \leqslant 1$, consider the splitting of $U(z) = X$ such that

$$U^1(z) = \left\{(x, y) \in X:\ y \leqslant |x - t|\right\},$$
$$U^2(z) = \left\{(x, y) \in X:\ y > |x - t|\right\}.$$

Then the corresponding complementary topologies $\sigma_1$ and $\sigma_2$ are the well known ones: $\sigma_1$ is the "Bow-tie" topology and $\sigma_2$ is the "Niemytzki" topology. The space $(X^\pm, \sigma_1 \bigoplus \sigma_2)$ is compact since $(X, \sigma)$ is compact.

Now we use the spaces in Sections 2 and 3 to make new spaces of the form $(X^\pm, \sigma_1 \bigoplus \sigma_2)$. Let $X = p^\omega$ and let $A_1$ and $A_2$ be disjoint subsets of $\{1, 2, \ldots, p-1\}$. Consider the topologies on $X$ such that $\sigma_1 = \tau_p(A_1)$, $\sigma_2 = \tau_p(A_2)$ and $\sigma = \tau_p(A_1 \cup A_2)$. (Let us suppose that $\tau_p(\emptyset)$ means the discrete topology.) Then $\sigma_1$ and $\sigma_2$ are complementary with respect to $\sigma$, and we get the space $((p^\omega)^\pm, \sigma_1 \bigoplus \sigma_2)$ and the 2–1 map $((p^\omega)^\pm, \sigma_1 \bigoplus \sigma_2) \to (p^\omega, \sigma)$. In case $A_1 \cup A_2 = \{1, 2, \ldots, p-1\}$, the topology $\sigma = \tau_p$ is compact, and hence the space $((p^\omega)^\pm, \sigma_1 \bigoplus \sigma_2)$ is also compact. Especially, if $A_1 = \{1, 2, \ldots, q\}$ where $q < p - 1$, we have

$$\sigma_1 = \tau_{p,q} \quad \text{and} \quad \sigma_2 = \tau_p(\{q+1, \ldots, p-1\});$$

since this $\sigma_2$ is identical with $\tau_{p,p-q-1}$ as noted before in Section 3, we denote $\sigma_1 \bigoplus \sigma_2$ by $\tau_{p,q} \bigoplus \tau_{p,p-q-1}$ and denote the duplicate space $((p^\omega)^\pm, \tau_{p,p-q-1})$ simply by $D_{p,q}$. From 2.8 we know that $\tau_{p,q}$ and $\tau_{p,p-q-1}$ are both monotonically normal iff $q = (p-1)/2$. So, in case $q = (p-1)/2$, the space $D_{p,q}$ is a union of two monotonically normal spaces; but unfortunately this space itself is not monotonically normal as the following shows.

**Proposition 4.4.** *Let $1 \leqslant q < p - 1$. Every monotone operator on $D_{p,q}$ has a 2-cycle.*

**Proof.** Put $X = p^\omega$ and let $G$ be an arbitrary monotone operator on $D_{p,q} = ((p^\omega)^\pm, \tau_{p,q} \bigoplus \tau_{p,p-q-1})$. We show that $G$ has a 2-cycle. Let $x \in X$. For simplicity, we denote $L_A(x; s)$ and $L_A(x)$, where $A = \{q+1, \ldots, p-1\}$, by $L_\star(x; s)$ and $L_\star(x)$ respectively. By the definition of the topology $\tau_{p,q} \bigoplus \tau_{p,p-q-1}$ the neighborhood bases at $x^+$ and $x^-$ consist of the sets of the form

$$W(x^+; L_q(x; s)) \quad \text{and} \quad W(x^-; L_\star(x; s))$$

respectively. Since the sets $G(x^+, W(x^+; L_q(x)))$ and $G(x^-, W(x^-; L_\star(x)))$ are open neighborhoods of $x^+$ and $x^-$ respectively, we can choose $\varphi(x) \in p^{<\omega}$ such that

$$W(x^+; L_q(x; \varphi(x))) \subseteq G(x^+; W(x^+; L_q(x))) \quad \text{and}$$
$$W(x^-; L_\star(x; \varphi(x))) \subseteq G(x^-; W(x^-; L_\star(x))).$$

So we get a mapping $\varphi : X \to p^{<\omega}$. Then by the same argument as in 2.5, we can find $u \subsetneqq t \in p^{<\omega}$ such that $B(t) \subseteq \operatorname{cl} \varphi^{-1}(u) \subseteq B(u)$ where cl is the closure with respect to the Cantor topology $\tau_p$ of $X$. Choose two points $y, z \in X$ such that

$$y \in B(t\hat{\ }0) \cap \varphi^{-1}(u) \quad \text{and} \quad z \in B(t\hat{\ }(p-1)) \cap \varphi^{-1}(u).$$

It suffices to show that $\{y^+, z^-\}$ is a 2-cycle for $G$. Since $(p-1) \dot{+} (q+1) = q$, we have $B(t\hat{\ }q) \subseteq L_q(t\hat{\ }0) \cap L_\star(t\hat{\ }(p-1))$, and so, $B(t\hat{\ }q) \subseteq L_q(y; u) \cap L_\star(z; u)$. Hence

$$B(t\hat{\ }q)^\pm \subseteq W(y^+; L_q(y; u)) \cap W(z^-; L_\star(z; u)).$$

Since $\varphi(y) = \varphi(z) = u$, we get

$$B(t\hat{\ }q)^\pm \subseteq G(y^+, W(y^+; L_q(y))) \cap G(z^-, W(z^-; L_\star(z))).$$

Note that $z \notin L_q(y)$ and $y \notin L_\star(z)$, and so,

$$z^- \notin W(y^+; L_q(y)) \quad \text{and} \quad y^+ \notin W(z^-; L_\star(z)).$$

Hence the monotonicity of $G$ implies

$$G(y^+, W(y^+; L_q(y))) \subseteq G(y^+, X^\pm \setminus \{z^-\}) \quad \text{and}$$
$$G(z^-, W(z^-; L_\star(z))) \subseteq G(z^-, X^\pm \setminus \{y^+\}).$$

Thus we get

$$B(t\hat{\ }q)^{\pm} \subseteq G(y^+, X^{\pm} \setminus \{z^-\}) \cap G(z^-, X^{\pm} \setminus \{y^+\})$$

which proves that $\{y^+, z^-\}$ is a 2-cycle for $G$. $\square$

Let us summarize the properties of the space $D_{p,q}$.

**Theorem 4.5.** *Let $1 \leqslant q < p-1$. Then the duplicate space*

$$D_{p,q} = \left((p^{\omega})^{\pm}, \tau_{p,q} \bigoplus \tau_{p,p-q-1}\right)$$

*is a 0-dimensional, separable, 1st countable compact Hausdorff space admitting a 2–1 perfect map onto the Cantor space $(p^{\omega}, \tau_p)$. Though this $D_{p,q}$ is not monotonically normal, in the special case $q = (p-1)/2$, it becomes a perfectly normal (equivalently, hereditary Lindelöf) homogeneous space.*

**Proof.** We consider the case $q = (p-1)/2$. Then the topology on $D_{p,q}$ becomes $\tau_{p,q} \bigoplus \tau_{p,q}$, and hence, is homogeneous. Since $\tau_{p,q}$ is monotonically normal by 2.8, $D_{p,q}$ is the union of two monotonically normal spaces. As noted before 1.3, any ccc monotonically normal space is hereditary Lindelöf [13]. Hence $D_{p,q}$ is hereditary Lindelöf. $\square$

Considering the special case $p \leqslant 10$, we get perfectly normal compact spaces $D_{3,1}$, $D_{5,2}$, $D_{7,3}$ and $D_{9,4}$ each of which is a union of two monotonically normal spaces. We don't know if these spaces are topologically distinct; for instance, $D_{3,1} \approx D_{5,2}$? It should be recalled that we don't even know if $\tau_{3,1} \approx \tau_{5,2}$; see Remark 2.10. Let $L$ be the lexicographically ordered compact space in Example 4.2, let $C$ denote the Cantor space, and define $\mathbb{C}$ to be a class of all spaces which are continuous images of some closed subspaces of $L \times C$. D.H. Fremlin asked (cf. [7]) if it is consistent with ZFC that every perfectly normal compact space belongs to $\mathbb{C}$, and Watson and Weiss [18] gave a ZFC counterexample. We don't know if our perfectly normal compacta $D_{p,q}$ where $q = (p-1)/2$ belong to the class $\mathbb{C}$; we conjecture they don't.

## 5. Another topology on the Sierpinski gasket

Let us consider the Sierpinski gasket (or Sierpinski triangle) $S$ in the plane $\mathbb{R}^2$, which is defined to be a self-similar set in the complex plane determined by the three maps $f_0$, $f_1$ and $f_2$ such that $f_0(z) = z/2$, $f_1(z) = (z+1)/2$ and $f_2(z) = z/2 + (1 + \mathrm{i}\sqrt{3})/4$. Let $\pi : 3^{\omega} \to S$ be the canonical, at most 2–1 map onto $S$. Let $\sigma_{3,1}$ be the quotient topology on $S$ obtained from $(3^{\omega}, \tau_{3,1})$ by the map $\pi$. Note that $|\pi^{-1}(x)| \leqslant 2$ for every $x \in S$ and that $|\pi^{-1}(x)| = 2$ for only countably many $x$'s. For $x \in S$ let $L[x; n]$ denote the set

$$\pi\Big[\bigcup\{L(y; y \upharpoonright n) \colon y \in \pi^{-1}(x)\}\Big].$$

Using Arhangel'skii's terminology [1] we can say that all sets of the form $L[x;n]$ ($x \in S$, $n \in \omega$) form a "weak base" for the topology $\sigma_{3,1}$. Put $\mathbb{F}(x) = \{L[x;n]\colon n \in \omega\}$ and $\mathbb{F} = \langle \mathbb{F}(x)\colon x \in S\rangle$. Then an open base for $\sigma_{3,1}$ is described as follows: Let $\varphi$ be an arbitrary section of $\mathbb{F}$, that is, $\varphi(x) \in \mathbb{F}(x)$ for each $x \in S = \operatorname{dom}(\varphi)$. Define inductively

$$\Sigma_0(x,\varphi) = \{x\}, \quad \Sigma_n(x,\varphi) = \bigcup\{\varphi(y)\colon y \in \Sigma_{n-1}(x,\varphi)\}$$

and put $\Sigma(x,\varphi) = \bigcup_{n\in\omega} \Sigma_n(x,\varphi)$. Then the sets of the form $\Sigma(x,\varphi)$ where $x \in S$, and $\varphi$ ranges over all sections of $\mathbb{F}$, become a base for $\sigma_{3,1}$. Notice that our topology $\sigma_{3,1}$ is Hausdorff, since it is finer than the Cantor topology $\tau_3$. But K. Eda [5] recently proved that $\sigma_{3,1}$ is not regular. Though this fact seems to be a slight defect, this nonregular, Hausdorff topology $\sigma_{3,1}$ looks quite natural as a new topology on the Sierpinski gasket. By a similar way we can consider quotient topologies $\sigma_{8,q}$ on the Sierpinski carpet induced from $(8^\omega, \tau_{8,q})$ where $1 \leqslant q \leqslant 6$, while, quotient topologies $\sigma_{20,q}$ on the Menger sponge induced from $(20^\omega, \tau_{20,q})$ where $1 \leqslant q \leqslant 18$. Investigation of these new topologies will be our future task.

## References


[1] A.V. Arhangel'skii, Mappings and spaces, Russian Math. Surveys 21 (1966) 115–162.

[2] D.K. Burke and E.K. van Douwen, No dense metrizable $G_\delta$-subspaces in Butterfly semi-metrizable Baire spaces, Topology Appl. 11 (1980) 31–36.

[3] W.W. Comfort, K.H. Hofmann and D. Remus, Topological groups and semigroups, in: M. Husek and J. van Mill, eds., Recent Progress in General Topology (Elsevier, 1992) 57–144.

[4] W.W. Comfort, A. Kato and S. Shelah, Topological partition relation of the form $\omega^* \to (Y)^1_2$, in: Papers on General Topology and Applications, Ann. New York Acad. Sci. 704 (1993) 70–79.

[5] K. Eda, Personal communication.

[6] G. Gruenhage, Generalized metric spaces, in: K. Kunen and J.E. Vaughan, eds., Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984) 423–501.

[7] G. Gruenhage, Perfectly normal compacta, cosmic spaces and some partition problems, in: J. van Mill and G.M. Reed, eds., Open Problems in Topology (North-Holland, Amsterdam, 1990) 85–95.

[8] G. Gruenhage, Generalized metric spaces and metrization, in: M. Husek and J. van Mill, eds., Recent Progress in General Topology (Elsevier, 1992) 239–274.

[9] R.W. Heath, D. Lutzer and P. Zenor, Monotonically normal spaces, Trans. Amer. Math. Soc. 178 (1973) 481–493.

[10] A. Kato, A new construction of extremally disconnected topologies, Topology Appl. 58 (1994) 1–16.

[11] P.J. Moody and A.W. Roscoe, Acyclic monotone normality, Topology Appl. 47 (1992) 53–67.

[12] D.B. Motorov, Homogeneity and $\pi$-networks, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 44 (1989) 31–34.

[13] A.J. Ostaszewski, Monotone normality and $G_\delta$-diagonals in the class of inductively generated spaces, Colloq. Math. Soc. János Bolyai 23 (1978) 905–930.

[14] M.E. Rudin, A cyclic monotonically normal space which is not $K_0$, Proc. Amer. Math. Soc. 119 (1993) 303–307.

[15] W.A.F. Ruppert, Compact semitopological semigroups, in: K.H. Hofmann, J.D. Lawson and J.S. Pym, eds., The Analytical and Topological Theory of Semigroups (Walter de Gruyter, 1990) 133–170.
[16] L.A. Steen and J.A. Seebach Jr, Counterexamples in Topology (Springer, Berlin, 1978), second edition.
[17] T. Terada, Spaces whose all nonempty clopen subspaces are homeomorphic, Yokohama Math. J. 40 (1993) 87–93.
[18] S. Watson and W. Weiss, A topology on the union of the double arrow space and the integers, Topology Appl. 28 (1988) 177–179.
[19] S. Watson, The construction of topological spaces, in: M. Husek and J. van Mill, eds., Recent Progress in General Topology (Elsevier, 1992) 673–757.